Chapter 6: Polynomials

POLYNOMIALS

Definition: A polynomial is an algebraic expression that is a sum of terms, where each term contains only variables with whole number exponents and integer coefficients.

Example: The following expressions are all considered polynomials:

$x^2 + 2x - 7$

$x^4 - 7x^3$

x

When we write a polynomial we follow the convention that says we write the terms in order of descending powers, from left to right.

The following are NOT polynomials:

$\frac{1}{x}$

$\sqrt{x^3 - 4}$

$x^2 + 3x + 2x^{-2}$

A polynomial can have any number of terms ("poly" means "many"). We have special names for polynomials that have one, two, or three terms:

MONOMIAL

A monomial has one term ("mono" means "one"). The following are monomials:

x

$3x^4$

$2x^3$

BINOMIAL

A binomial has two terms:

$x + 1$

$5x^2 - 3x$

TRINOMIAL

A trinomial has three terms:

$x^4 + 2x^3 - 3x$

$2x^2 - 4x + 1$

DEGREE OF A TERM

The *degree* of an individual term in a polynomial is the sum of powers of all the variables in that term. We only have to use the plurals in this definition because of the possibility that there may be more than one variable. In practice, you will most often see polynomials that have only one variable (traditionally denoted by the letter '*x*'). In that case, the degree will simply be the power of the variable.

Examples:

$2x^3$	Degree = 3
$3x^4$	Degree = 4
x	Degree = 1
$3x^2y^5$	Degree = 7 (because 2 + 5 = 7)
37	Degree = 0

Why is the last example, which is just a plain number, considered to be of degree zero? It is because of the fact that $x^0 = 1$, and everything has a factor of 1. So we can say that 37 is the coefficient of x^0.

DEGREE OF A POLYNOMIAL

The degree of the entire polynomial is the degree of the highest-degree term that it contains, so

$x^2 + 2x - 7$ is a second-degree trinomial, and

$x^4 - 7x^3$ is a fourth-degree binomial. Addition and Subtraction of Polynomials

Adding (or subtracting) polynomials is really just an exercise in collecting like terms. For example, if we want to add the polynomial

$2x^2 + 4x - 3$

to the polynomial

$6x + 4$,

we would just put them together and collect like terms:

$$\begin{aligned}(2x^2 + 4x - 3) + (6x - 4) &= 2x^2 + 4x - 3 + 6x - 4 \\ &= 2x^2 + 10x - 7\end{aligned}$$

Notice that the parentheses in the first line are only there to distinguish the two polynomials.

Although this is basically just a bookkeeping activity, it can get a little messy when there are many terms. One way to help keep things straight is use the column format for addition, keeping like terms lined up in columns:

$$\begin{array}{rrr} 2x^2 & + \; 4x & -3 \\ + \qquad & 6x & -4 \\ \hline 2x^2 & +10x & -7 \end{array}$$

This method is particularly helpful in the case of subtraction, because it is too easy to make a mistake distributing the minus sign when you write it all in one row.

MULTIPLICATION OF POLYNOMIALS

- The general rule is that **each term in the first factor has to multiply each term in the other factor**
- The number of products you get has to be the number of terms in the first factor times the number of terms in the second factor. For example, a binomial times a binomial gives four products, while a binomial times a trinomial gives six products.
- Be very careful and methodical to avoid missing any terms
- After the multiplication is complete you can try to collect like terms to simplify the result

EXAMPLE: PRODUCT OF A BINOMIAL AND A TRINOMIAL

$(x + 2)(x^2 - 2x + 3)$

There are six possible products. We can start with the x and multiply it by all three terms in the other factor, and then do the same with the 2. It would look like this:

$(x + 2)(x^2 - 2x + 3)$

$= (x)x^2 - (x)2x + (x)3 + (2)x^2 - (2)2x + (2)3$

$= x^3 - 2x^2 + 3x + 2x^2 - 4x + 6$

$= x^3 - x + 6$

This method can get hard to keep track of when there are many terms. There is, however, a more systematic method based on the stacked method of multiplying numbers:

Stack the factors, keeping like degree terms lined up vertically:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline \end{array}$$

Multiply the 2 and the 3:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline 6 \end{array}$$

Multiply the 2 and the –2x:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline -4x + 6 \end{array}$$

Multiply the 2 and the x^2:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline 2x^2 - 4x + 6 \end{array}$$

Now multiply the x by each term above it, and write the results down underneath, keeping like degree terms lined up vertically:

$$\begin{array}{rrr} x^2 & - 2x & + 3 \\ & x & + 2 \\ \hline 2x^2 & - 4x & + 6 \\ & 3x & \end{array}$$

$$\begin{array}{rrrr} & x^2 & - 2x & + 3 \\ & & x & + 2 \\ \hline & 2x^2 & - 4x & + 6 \\ -2x^2 & + 3x & & \end{array}$$

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline 2x^2 - 4x + 6 \\ x^3 - 2x^2 + 3x \end{array}$$

Then you just add up the like terms that are conveniently stacked above one another:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline 2x^2 - 4x + 6 \\ x^3 - 2x^2 + 3x \\ \hline x^3 + 0 \quad - \quad x + 6 \end{array}$$

This stacked method is much safer, because you are far less likely to accidentally overlook one of the products, but it does take up more space on the paper.

Product of a Monomial and a Binomial: Distributive Law

Example:

$ab(2a + 1) = ab(2a) + ab(1) = 2a^2b + ab$

Product of Two Binomials: FOIL (First-Outer-Inner-Last)

Because the situation of a binomial times a binomial is so common, it helps to use a quick mnemonic device to help remember all the products. This is called the FOIL method.

Example:

$$(x + 2)(x + 3)$$

F O I L

$$x^2 + 3x + 2x + 6$$

1. The F stands for *first*, which means the x in the first factor times the x in the second factor
2. The O stands for *outer*, which means the x in the first factor times the 3 in the second factor